SECTIONS OF POINT SETS*

BY

DEANE MONTGOMERY

1. Introduction

A section of a plane point set E is defined as that subset of E which contains all points of E lying on a line L. If L is a horizontal line the section is called a horizontal section and if L is a vertical line, the section is called a vertical section. It is the purpose of this paper to study the relations between E and its horizontal and vertical sections. Kuratowski and Ulam†, Sierpinski‡, and Fubini§, have considered various phases of this problem. Baire‖, Hahn¶, Kempisty** and others have considered the closely related problem of finding the relations between a function $f(x, y)$ and the functions obtained by holding x or y constant.

In order to state results in a general manner, E will be regarded as a subset of a combinatorial product space $A\times B$ where A and B are metric spaces and B is separable. Such a space is defined as the collection of all pairs of points (x, y), x being a point of A and y being a point of B. The distance between (x_1, y_1) and (x_2, y_2) is here defined to be $[(x_1x_2)^2+(y_1y_2)^2]^{1/2}$.†† The plane is a special case of such a space in which A and B are straight lines, and all the results of this paper apply to the plane and also to an $(m+n)$-dimensional euclidean space considered as the product of an m-dimensional and an n-dimensional euclidean space.

Because $A\times B$ is analogous to the plane, the subset of points (x, y) such that $x=a$ is called a vertical section of $A\times B$ and is denoted by $a\times B$ or $(x=a)$; similarly the subset of points (x, y) such that $y=b$ is called a horizontal section of $A\times B$ and is denoted by $A\times b$ or $(y=b)$. If E is any subset of $A\times B$ the set $E\cdot(x=a)$ is called a vertical section of E and the set $E\cdot(y=b)$ is called a horizontal section of E.

---

* Presented to the Society, November 25, 1932; received by the editors February 1, 1933.

† Fundamenta Mathematicae, vol. 19, p. 247; see also an article by Kuratowski in vol. 17, p. 275.

‡ Fundamenta Mathematicae, vol. 1, p. 112.

§ Rendiconti della Reale Accademia dei Lincei, (5), vol. 16, I. For a statement of Fubini's theorem see also Carathéodory, *Vorlesungen über Reelle Funktionen*, 1927, p. 621.

‖ Annali di Matematica, 1899, p. 1.

¶ Mathematische Zeitschrift, 1919, p. 306.

** Fundamenta Mathematicae, vol. 14, p. 237, and vol. 19, p. 184.

†† If p and q are any two points of a metric space, (pq) denotes the distance between p and q.

If E is closed, all horizontal and vertical sections of E are closed, and if E is open, its horizontal and vertical sections are open (relative to the sections of $A \times B$ which contain them). A similar proposition is true for sets F or O of type α. Converse propositions are not true. The plane set of points $(1/n, 1/n)$, where n takes all integral values, is such that each of its horizontal and vertical sections contains at most one point and is therefore closed. The point $(0, 0)$ is a limit point of the set which is not in the set. Sierpinski* has constructed a plane set every section of which (not merely horizontal and vertical sections) contains at most two points and which is non-measurable in the Lebesgue sense. This example shows that the fact that every horizontal and vertical section of E is of type α is not a sufficient condition that E be of type α, and that in order to obtain such a sufficient condition, further restrictions on the sections or on the relations between them must be imposed. By restricting the vertical sections to a type of set called I-set (or the complement of such a set) sufficient conditions may be obtained that a set be of various types. This is done in §3. Necessary and sufficient conditions that sets with restricted vertical sections be of class α are given in §6. Uses of sets called gratings are considered in §7. Theorems are given which show that boundaries of sets with certain kinds of sections lie on sets of lines of the first category. The results are applied in §8 to prove a theorem of Baire concerning functions of two variables continuous in each of them and to obtain a result regarding Kempisty's generalization of this theorem.

2. Horizontal sections of class M

The following definitions will be useful.

Definition 1. *If the inner points of a set are dense on the set, the set is called an I-set.*

A set may have this property with respect to $A \times B$ or it may be a subset of a section of $A \times B$ and have this property with respect to the section, this latter being the case which will most often arise.

Definition 2. *Given a point* (a, b), *the set of points* (a, y) *where* $(by) < r$, r *a positive number, is called an open vertical interval of center* (a, b) *and radius* r.

A closed vertical interval is defined in the same way except that $(by) \leqq r$. Closed and open horizontal intervals may also be defined. A vertical interval might also be defined as $a \times S$ where S is a sphere in B of center b and radius r.

* See the previous reference.

DEFINITION 3. *If a set G lies on a horizontal section, $G(r)$ is the set of points of open vertical intervals of radii r and centers at the points of G.*

DEFINITION 4. *If $\mathfrak{M}$ is a family* of point sets M lying on horizontal sections of $A\times B$, $\mathfrak{M}(r)$ is the family of all point sets $M(r)$ for r ranging over all positive numbers.*

If $\mathfrak{M}$ is a family of point sets, $\mathfrak{M}_\sigma(\mathfrak{M}_\delta)$ denote, as is conventional, the families composed of all possible sums (products) of an enumerable number of sets of $\mathfrak{M}$.

A set of vertical sections K is said to be everywhere dense if the set of points $[a]$, such that $(x=a)$ is in K, is everywhere dense in A. In a similar manner other point-set properties, for example the property of being in the first or the second category, are ascribed to sets of vertical sections and to sets of horizontal sections as well.

By the projection of a point (x, y) on $(y=b)$ is meant the point (x, b). The projection of a set of points E on $(y=b)$ is the set of points formed by projecting all the points of E on $(y=b)$.

THEOREM 1. *If each vertical section of E is an open set whose complement is an I-set† and horizontal sections of E belong to $\mathfrak{M}$, then E belongs to the family $[\mathfrak{M}_\delta(r)]_\sigma$.*

Since B is separable there exists an enumerable everywhere dense set $(y=b_i)$ of horizontal sections of $A\times B$. Let r_i be a sequence of positive numbers approaching 0 as a limit. Let K_j be the projection of $E\cdot(y=b_j)$ on $(y=b_i)$. Let A_{im} be the product of $E\cdot(y=b_i)$ and all sets K_j such that $(b_jb_i)<r_m$. The set $A_{im}(r_m)$ is a subset of E. To prove this, suppose that $A_{im}(r_m)$ contains a point (a, b) of CE. The point (a, b_i) is in $A_{im}(r_m)$ and also all points of the open vertical interval of radius r_m and center (a, b_i) are in $A_{im}(r_m)$. Since (a, b) lies in this open vertical interval, there is by hypothesis some inner point‡ (a, c) of CE in this interval. There is then an ϵ such that if $(cy)<\epsilon$, (a, y) is in CE. Since the b_i's are everywhere dense in B there is some b_i, say b_n, such that $(cb_n)<\epsilon$. The point (a, b_n) is then in CE. This b_n may be so chosen that $(b_ib_n)<r_m$. The set $E\cdot(y=b_n)$ does not con-

* It is here supposed that if E consisting of points (x, b) is in $\mathfrak{M}$, the set of points (x, c), where x has the same range as in E and c is any point of B, is also in $\mathfrak{M}$. This restriction is made for convenience. It is necessary for the proofs of some of the following theorems in which use is made of the projections of sets from one horizontal section to another.

† A more explicit statement is as follows: If V is any vertical section of $A\times B$, $V\cdot E$ is open in V and $V-E$ is an I-set in V, etc. Language similar to that in the hypothesis of Theorem 1 will be used throughout, with the meaning given in this note.

‡ With respect to the section $(x=a)$.

tain (a, b_n) and K_n will not contain (a, b_i). Therefore (a, b_i) will not be in $A_{im}(r_m)$. From this contradiction, it follows that $A_{im}(r_m)$ is in E.

To prove that every point of E is in some $A_{im}(r_m)$, let (c, d) be any point of E. There is an ϵ such that if $(dy)<\epsilon$, (c, y) is in E. Choose r_m and b_i such that $b_id<r_m<\epsilon/2$. Every point (c, y) of the open vertical interval of center (c, b_i) and radius r_m is in E. For since $(b_iy)<\epsilon/2$ and $(b_id)<\epsilon/2$, it follows from the triangle axiom that $(dy)<\epsilon$. Therefore the set K_j contains (a, b_i) if $(b_ib_j)<r_m$, and $A_{im}(r_m)$ contains all points (c, y) such that $(b_iy)<r_m$. It must then contain (c, d) since $(b_id)<r_m$.

Thus $E=\sum_{im}A_{im}(r_m)$, which proves the theorem.

THEOREM 2. *If vertical sections of E are closed I-sets and horizontal sections of E belong to the family* $\mathfrak{M}$, *then E belongs to the family* $[\mathfrak{M}_\sigma(r)]_{\sigma\delta}$.

Let $(y=b_i)$ be an enumerable everywhere dense set of horizontal sections of $A\times B$ and let (r_i) be a sequence of positive numbers approaching 0. Let K_j be the projection of $E\cdot(y=b_j)$ on $(y=b_i)$. Let A_{im} be the sum of $E\cdot(y=b_i)$ and all sets K_j such that $(b_ib_j)<r_m$. The set $E_m=\sum_iA_{im}(r_m)$ is a member of the family $[\mathfrak{M}_\sigma(r)]_\sigma$. The set E_m contains E, for all m. To prove this let (c, d) be any point of E. By hypothesis there is for each r_m a vertical interval V_e, of radius e, which contains only points of E and for every point (c, y) of which $(dy)<r_m$. There is some $(y=b_i)$ which cuts V_e in a point (c, b_i). The point (c, b_i) is then in E. The set $A_{im}(r_m)$ contains (c, d) because $(db_i)<r_m$.

The set E is therefore included in $\prod_mE_m$. To prove that $E=\prod_mE_m$, let (a, b) be any point of CE. There is an ϵ such that if $(by)<\epsilon$, (a, y) is in CE. In order that (a, b) be in $A_{im}(r_m)$ it is necessary that $(bb_i)<r_m$. Choose $r_m<\epsilon/2$. If $(bb_i)<r_m$, and $(b_iy)<r_m$, it follows that $(by)<\epsilon$, and consequently (a, y) is in CE. Therefore if $(bb_i)<r_m$, $A_{im}(r_m)$ cannot contain (a, b) because all points on the open vertical interval of center (a, b_i) and radius r_m are in CE. As has been mentioned, $A_{im}(r_m)$ cannot contain (a, b) if $(bb_i)\geqq r_m$. Therefore for r_m chosen as it has been, (a, b) is not in E_m and consequently not in $\prod_mE_m$. It follows that $E=\prod_mE_m$, which completes the proof of the theorem.

3. HORIZONTAL SECTIONS OF THE TYPE α

LEMMA 1. *Let G be a set lying on a horizontal section of* $A\times B$. *If G is an* $O_\alpha(\alpha\geqq 0)$ *or an* $F_\alpha(\alpha>0)$, $G(r)$ *is an* O_α *or an* F_α.* *If G is analytic*, $G(r)$ *is analytic.*

The first part of the lemma is true for sets O_0 and sets O_1, and can be shown to be true for sets O_α by transfinite induction. The latter part of the lemma may be readily proved from the definition of an analytic set.

* For a discussion of sets F_α and O_α, see de la Vallée Poussin, *Intégrales de Lebesgue*, p. 132.

THEOREM 3. *If the horizontal sections of E are O_α's and the vertical sections are closed I-sets, then E is an $F_{\alpha+1}$.*

This follows from Theorem 2. $\mathfrak{M}$ is here the family of O_α's in horizontal sections of $A\times B$. $\mathfrak{M}_\sigma$ is the same family and by the preceding lemma $\mathfrak{M}_\sigma(r)$ is a family of O_α's in $A\times B$, as is $[\mathfrak{M}_\sigma(r)]_\sigma$. Therefore $[\mathfrak{M}_\sigma(r)]_{\sigma\delta}$ is a family of $F_{\alpha+1}$'s in $A\times B$ and E must be an $F_{\alpha+1}$.

By taking complements the following is proved:

THEOREM 4. *If the horizontal sections of E are F_α's and the vertical sections are open sets whose complements are I-sets, then E is an $O_{\alpha+1}$.*

That the change in classification mentioned in Theorems 3 and 4 actually may occur, is shown by the following plane set. On the line $y=x$, take a set E which is an $O_{\alpha+1}(\alpha\geqq 1)$* and an O of no lower class.† Then $E=\sum_i A_i$ where A_i is an F_α at most. At each point of A_i erect a vertical interval of length $1/i$, closed at the end touching $y=x$ and open at the other end, and denote the set thus obtained by H. Horizontal sections of H are F_α's. This can be seen as follows. If L is any horizontal line and p any point on this line above the line $y=x$, vertical intervals from only a finite number of the sets A_i can cut L to the left of, or at, p. Therefore the points of H on L to the left of or at p must form an F_α. This is true however close p may be to $y=x$. Let p_n be a sequence of points on L above $y=x$, approaching $y=x$. Let E_n be the points of $(CH)\cdot L$ to the left of or at p_n. Then E_n is an O_α, and $\sum E_n$ is an O_α. Therefore the points of CH on L to the left of $y=x$ form an O_α and consequently the points of H on L form an F_α. Since $\alpha\geqq 1$, it does not matter whether or not the intersection of L and $y=x$ is in H. Although horizontal sections of H are F_α's, the set H itself must be at least an $O_{\alpha+1}$ since $y=x$ cuts it in an $O_{\alpha+1}$.

Denote by R that part of the complement of H which lies on or above $y=x$. Horizontal sections of R are O_α's but R itself is an $F_{\alpha+1}$ at least, since $y=x$ cuts it in an $F_{\alpha+1}$. By Theorem 3, R is an $F_{\alpha+1}$ at most. This example shows that under the hypothesis of Theorem 3, it is impossible to draw a stronger conclusion on the F classification of E than the one there given.

THEOREM 5. *If the horizontal sections of E are O_α's and the vertical sections are open sets whose complements are I-sets, then E is an $O_{\alpha+2}$.*

This follows from Theorem 1. The family $\mathfrak{M}$ is here the family of sets O_α on horizontal sections of $A\times B$. $\mathfrak{M}_\delta$ is then the family of sets $F_{\alpha+1}$ on hori-

* A single open vertical interval furnishes an example in case $\alpha=0$.

† For a proof of the existence of functions of all classes (which proves the existence of sets of all classes) see de la Vallée Poussin, *Intégrales de Lebesgue*, p. 145 ff.

zontal sections. $\mathfrak{M}_\delta(r)$ contains only sets $F_{\alpha+1}$ in $A \times B$ by Lemma 1. Therefore $[\mathfrak{M}_\delta(r)]_\sigma$ contains only sets $O_{\alpha+2}$.

By taking complements there is proved

THEOREM 6. *If the horizontal sections of E are F_α's and the vertical sections are closed I-sets, then E is an $F_{\alpha+2}$.*

Whether or not the classification may actually be increased by two under the hypothesis of Theorems 5 and 6 is an open question. That an advance of one may occur is shown by the following plane set. Construct the set H as in the preceding example, except that the vertical intervals are now to be closed instead of half closed. Horizontal sections of this set are F_α's as before and the set itself must be an $O_{\alpha+1}$ exactly as H was.

The two following theorems may be proved directly or as a result of the preceding theorems on sets O_α and F_α.

THEOREM 7. *If horizontal sections of E are A_α's *and vertical sections of E are all closed I-sets or all open sets whose complements are I-sets, then E is an $A_{\alpha+2}$.*

4. Analytic or measurable horizontal sections

If $\mathfrak{M}$ is the family of analytic sets, $\mathfrak{M}_\sigma$ and $\mathfrak{M}_\delta$ are the same family, from which we have the following theorem.

THEOREM 8. *If horizontal sections of E are analytic and vertical seciions of E are all closed I-sets or all open sets whose complements are I-sets, E is analytic.*

If there is a theory of measure in the space under consideration, as for example in the plane, a theorem similar to Theorem 8 is true for measurable sets.

5. The set E_e

Let E_e denote the subset of E each point of which lies on a closed vertical interval of radius exactly e, which contains only points of E.

For the two theorems of this section the conditions on A and B are that they are metric, that every closed sphere in B is compact and that inner points of a closed sphere in B are dense on the sphere.

THEOREM 9. *If horizontal and vertical sections of E are closed, E_e is closed.*

Let (a_n, b_n) be a sequence of points of E_e converging to a limit point (a, b). Each (a_n, b_n) lies on a closed vertical interval, V_e^n, containing only points of E, of radius e, and of center (a_n, c_n). An infinite number of points b_n are such that $(bb_n) < \epsilon$, where ϵ is any positive number. The fact that $(b_n c_n) \leq e$ for all n

* For a discussion of sets A_α, see de la Vallée Poussin, *Intégrales de Lebesgue*, p. 135.

implies that for an infinite number of points c_n, $(bc_n) < e+\epsilon$, and by hypothesis, these points have some limit point c such that $(bc) \leqq e$. Let V_e be the closed vertical interval of center (a, c) and radius e. Let (a, y) be any point such that $(cy) < e$, that is, any point on the interior of the vertical interval V_e. It will now be shown that there is an n for which (a_n, y) is in E and $(a_n a) < \eta$, where η is any positive number. Consider the n's for which $(a_n a) < \eta$ and select from this group an n such that $(c_n c) < e-(cy)$, that is, such that $(c_n c)+(cy) < e$. It follows that $(c_n y) < e$ and therefore the point (a_n, y) is in E. On the horizontal section $(y=y)$ there is thus a sequence of points of E approaching (a, y), and since horizontal sections of E are closed, (a, y) must be in E. Therefore, all inner points of the vertical interval V_e are in E. Because of the hypothesis on the space B and the fact that vertical sections of E are closed, it follows that every point of V_e is in E. Since $(bc) \leqq e$, the point (a, b) is in V_e, which proves that every limit point of E_e is in E_e.

For Theorem 10, B is required to have the further property that any point p of B on a closed sphere of radius $>r$ is on a sphere of radius exactly r, which is contained in the first sphere.

THEOREM 10. *If horizontal and vertical sections of E are closed and each point of E lies on a closed vertical interval containing only points of e, then E is an F_σ.*

Let (r_i) be a sequence of positive numbers approaching 0. Any point in E_r is in E_{r_i} if $r_i \leqq r$. Hence $E=\sum E_{r_i}$ and since E_{r_i} is closed, E is an F_σ.

6. UNIFORMITY PROPERTIES

DEFINITION 5. *A point p is said to be a point of uniformity of E if, for some open sphere S in $A\times B$ of center p and for some e, $E\cdot S=E_e\cdot S$.*

DEFINITION 6. *A point p is said to be a point of uniform separation of E if it is a point of uniformity of CE.*

The points of uniformity of E form an open set.

For the theorems of this section, A and B are metric separable spaces. In addition it is required that every sphere in B be totally limited* and that B have the property stated just before Theorem 10.

THEOREM 11. *If the vertical sections of E are open, a necessary and sufficient condition that E be an O_α is that the set of points of non-uniform separation of E in E be an O_α and that horizontal sections of E be O_α's.*

The necessity will first be demonstrated. If E is an O_α its horizontal sections are O_α's. Let N denote the set of points of non-uniform separation of E.

* See Hausdorff, *Mengenlehre*, 1927, p. 108.

Since N is closed, it follows that $E\cdot N$ is an O_α if α is greater than 0. For $\alpha=0$, the necessity is obvious.

The sufficiency will now be demonstrated. Let p be any point of E which is a point of uniform separation of E. For some open sphere S of center p and some e, $S\cdot(CE)_e=S\cdot(CE)$. Let r_i be a sequence of positive numbers approaching 0 and let $(y=b_i)$ be an enumerable everywhere dense set of horizontal sections of $A\times B$. Let r_m be a fixed element of the sequence r_i and let ϵ be the smaller of the two numbers $e/4$ and $r_m/4$. Let S_1 be a sphere with the same center as that of S and with radius $2e$ larger than that of S. Because the projection of S_1 in B is a sphere in B it is totally limited in B by hypothesis. There exists then a finite set of horizontal sections $(y=h_i)$ such that every point of S_1 is a distance less than ϵ from some one of them. Let K_j be the projection of $S\cdot E\cdot(y=h_j)$ on $(y=b_i)$. Denote by A_{im} the product of $S\cdot E\cdot(y=b_i)$ and all K_j's such that $(h_jb_i)<r_m$.

In order to show that $A_{im}(r_m/2)$ is in $S\cdot E$, let (a, y) be any point of $A_{im}(r_m/2)$ and suppose that (a, y) is in $C(S\cdot E)$. It is then on a closed vertical interval V_e, of radius e, containing only points of $C(S\cdot E)$. By the hypothesis on the space B, the point (a, y) is also on a vertical interval V_ϵ, of radius exactly ϵ, V_ϵ being contained in V_e. The interval V_ϵ contains only points of $C(S\cdot E)$. Let (a, b) be the center of V_ϵ. Since (a, b) must be in S_1 there is some h_n such that $(bh_n)<\epsilon$. The point (a, h_n) is therefore in $C(S\cdot E)$. From the relations $(b_iy)<r_m/2$ and $(by)<\epsilon$, it follows that $(bb_i)<\frac{3}{4}r_m$. Since $(bh_n)<\epsilon$, it follows that $(h_nb_i)<\frac{3}{4}r_m+\epsilon<r_m$. The point (a, h_n), being in $C(S\cdot E)$, cannot be in $S\cdot E\cdot(y=h_n)$. Therefore (a, b_i) is not in K_n nor in A_{im}. Neither (a, b_i) nor (a, y) can then be in $A_{im}(r_m/2)$. From this contradiction it follows that $A_{im}(r_m/2)$ is in $S\cdot E$.

The proof showing that each point of $S\cdot E$ is in some $A_{im}(r_m/2)$ is analogous to the proof of a similar proposition given in the demonstration of Theorem 1, and will not be repeated here. Assuming this to be proved, $S\cdot E=\sum_{im}A_{im}(r_m/2)$. The set A_{im} is a finite product of O_α's and must be an O_α. Each $A_{im}(r_m/2)$ must then be an O_α, and therefore $S\cdot E$ is an O_α.

Every point of uniform separation of E is therefore the center of an open sphere S such that $S\cdot E$ is an O_α. By Lindelöf's* theorem an enumerable set (S_i) of such spheres cover the points of uniform separation of E in E. The set N of points of non-uniform separation of E in E is an O_α by hypothesis. Therefore $E=\sum_iS_i\cdot E+N$ is an O_α.

---

* This holds since $A\times B$ is separable when A and B are. See the previously cited paper by Kuratowski and Ulam.

THEOREM 12. *If the vertical sections of E are closed, a necessary and sufficient condition that E be an* $F_\alpha(\alpha > 0)$ *is that the set of points of non-uniformity of E in E be an* F_α *and that horizontal sections of E be* F_α*'s.*

The necessity will first be demonstrated. Horizontal sections of E are F_α's since they are the products of E and horizontal sections of the space $A \times B$. The set N of points of non-uniformity of E is closed and the product of N and E must be an F_α.

The sufficiency will now be shown. Let p be any point of uniformity of E. It is a point of uniform separation of CE and for some sphere S of center p, $(CE)\cdot S$ is an O_α by Theorem 11. As before, an enumerable number of such spheres, S_i, cover the points of uniformity of E. Since the points of CE in $\sum_i S_i$ are an O_α, the points of E in $\sum_i S_i$ form an F_α. The set N of points of non-uniformity of E in E form an F_α by hypothesis. Since $E = \sum_i E \cdot S_i + N$, E is an F_α.

The theorem is not true for $\alpha = 0$.

7. GRATINGS AND CATEGORICITY

In this section A and B are to be metric, separable, locally compact* spaces. In such spaces the complement of a set of the first category is of the second category, and open sets are of the second category. These propositions may be proved by a method similar to the method used for proving them in euclidean space. It is necessary to make use of the fact that every monotonic decreasing sequence of non-null compact spheres has a non-null product.†

DEFINITION 7. *If a horizontal section* $(y = b)$ *contains a set H of the second category* [*in* $(y = b)$], *such that* $H(r)$ *is in E for some r,* $(y = b)$ *is said to have property C with respect to E.*

DEFINITION 8. *If a horizontal section* $(y = b)$ *contains a set H, in and everywhere dense in a set O* [*in and open in* $(y = b)$] *such that* $H(r)$ *is in E for some r,* $(y = b)$ *is said to have property D with respect to E.*

The set $H(r)$ of Definition 8 is called a grating. A point p is said to be *within* the grating if p is in $O(r)$. A point p is *on* the grating if it is in $H(r)$.‡ Property C implies property D since a set of the second category must contain a subset everywhere dense in some open set.

LEMMA 2. *If* $\mathfrak{K}$, *a set of vertical sections, is of the second category and if each* $K \epsilon \mathfrak{K}$ *contains a vertical interval including only points of E, then some horizontal section has properties C and D with respect to E; furthermore the center of one of the vertical intervals is on the grating (of property D).*

* For a definition of this term see Fréchet, *Les Espaces Abstraits*, p. 223.

† See Banach, *Théorie des Opérations Linéaires*, pp. 13 and 14.

‡ If p is *on* a grating it is *within* the grating.

Let K_{3e} be the set of vertical sections containing vertical intervals of E of radii $>3e$, where e has been so chosen that K_{3e} is of the second category. Let V_{3e} denote an individual one of these vertical intervals of radius $>3e$ and let $\sum V_{3e}$ denote the points in all such intervals. From each V_{3e} form a vertical interval V_e of radius exactly e with the same center as V_{3e}. Each interval V_e consists entirely of points of E. Let (b_i) be an enumerable set of points everywhere dense in B, and let B_i be a sphere in B of center b_i and radius $2e$. Let A_i be the subset of A such that for each $a \epsilon A_i$, there is some V_e and a corresponding V_{3e} for which $V_e < a \times B_i < V_{3e}$.

It will now be shown that for each V_e and corresponding V_{3e}, of center (a, b), there is an i such that $V_e < a \times B_i < V_{3e}$, and consequently that $\sum A_i$ is the set in which the sections of K_{3e} cut A. In order to do this, choose i such that $(bb_i) < e$. For each point (a, y) of V_e, $(by) < e$. By the triangle axiom, $(b_i y) < 2e$. This shows that the vertical interval $a \times B_i$ of center (a, b_i) and radius $2e$ includes V_e. It will now be shown that $a \times B_i < V_{3e}$. If (a, y) is any point of $a \times B_i$, $(b_i y) < 2e$. Since $(bb_i) < e$, it follows from the triangle axiom that $(by) < 3e$ which is the condition that (a, y) be in V_{3e}.

Since $\sum A_i$ is the set in which the sections of K_{3e} cut A, it follows that $\sum A_i$ is of the second category and, consequently, that some particular A_i, say A_n, is of the second category. The section $(y = b_n)$ has property C with respect to E, for the set $A_n \times b_n$ is of the second category in $(y = b_n)$, and each point of $A_n \times b_n$ is the center of a vertical interval of radius $2e$ which includes only points of E. The section $(y = b_n)$ must then also have property D with respect to E. Since each of these vertical intervals contains an interval V_e it must contain the center of this interval V_e which is the center of the corresponding original interval V_{3e}. Therefore the center of one of the original intervals is on each vertical interval of the grating.

It is evident that when A and B have similar properties, the parts played by horizontal and vertical sections in any theorem may be interchanged.

Definition 9. *A point is said to be of the second category with respect to E if every neighborhood of the point contains a subset of E of the second category.*

The set of all points of the second category with respect to E is denoted by E_{sc}. The set E_{sc} is closed.

A necessary and sufficient condition that E be of the second category is that E_{sc} be of the second category. This implies that if E is of the second category, it must be of the second category at a set everywhere dense in an open set, and since E_{sc} is closed it must be of the second category at each point of an open set.†

† See Banach, *Théorie des Opérations Linéaires*, p. 13 and the reference there given.

THEOREM 13. *If vertical sections of R are I-sets and horizontal sections of T are I-sets, and $R\cdot T=0$, then $R'\cdot T+R\cdot T'$ is of the first category in $A\times B$.*

It is sufficient to show that $R\cdot T'$ is of the first category in $A\times B$. Assume that $R\cdot T'$ is of the second category in $A\times B$. It must then be of the second category at every point of a set O, open in $A\times B$, which implies that R and T are both dense in O. It follows from the hypothesis, that on each vertical section containing a point of $R\cdot T'\cdot O$, there is a vertical interval V containing only points of R and such that V is in O. Since the vertical sections containing points of $R\cdot T'\cdot O$ form a set of the second category†, Lemma 2 may be applied. By this lemma, there is a horizontal section L containing a set H everywhere dense in O^* (O^* a set open in L) such that $H(r)$ is in R. The set $H(r)$ is also in O since the intervals V from which it is constructed are in O. Suppose there is a point p of T within $H(r)$ and let the horizontal section containing p be L^*. The section L^* must contain an inner point (with respect to L^*) of T which lies in $O^*(r)$. But this is impossible because $L^*\cdot H(r)$ is dense in $L^*\cdot O^*(r)$ and $H(r)$ contains only points of R. There can be, then, no points of T within $H(r)$, but this is a contradiction since T must be dense in O. Therefore $R\cdot T'$ is of the first category in $A\times B$.

COROLLARY 1. *If vertical sections of E are I-sets and horizontal sections of CE are I-sets, then $E'\cdot(CE)+E\cdot(CE)'$ is of the first category in $A\times B$.*

This follows immediately from the theorem and the fact that $E\cdot(CE)=0$.

COROLLARY 2. *If vertical sections of E are I-sets and horizontal sections of CE are I-sets, there is an inner point of either E or CE in every set O, open in $A\times B$.*

This follows from Corollary 1. Points not belonging to $E'\cdot(CE)+E\cdot(CE)'$ are inner points either of E or of CE and this set is everywhere dense in $A\times B$.

THEOREM 14. *If horizontal and vertical sections of E are I-sets and horizontal sections of CE are I-sets, then E is an I-set in $A\times B$.*

Let p be any point of E lying on a horizontal section L and let O be any open set in $A\times B$ containing p. It is necessary to show that O contains an inner point of E. By hypothesis $L\cdot E$ contains a set O^* open in L. The set O^* is of the second category in L, and each vertical section K cutting O^* must contain a vertical interval V including only points of E and lying in O. From these V's, there may be formed a grating $H(r)$ containing only points of E and contained in O. No point of CE can be within this grating because hori-

† If this were not true, $R\cdot T'\cdot O$ would be of the first category in $A\times B$.

zontal sections of CE are I-sets. The argument is similar to the one in Theorem 13 and will not be repeated.

Kuratowski and Ulam have a theorem similar to the following:

THEOREM 15. *If E is a set whose horizontal sections are I-sets, and O is an open set in $A\times B$, a necessary and sufficient condition that $E\cdot O$ be dense in O, is that the vertical sections L for which $L\cdot E\cdot O$ is not dense in $L\cdot O$ form a set $\mathcal{L}$ of the first category.*

The sufficiency of the condition follows from the fact that if vertical sections K, such that $K\cdot E\cdot O$ is dense in $K\cdot O$, form a set complementary to a set of the first category, they are everywhere dense and therefore the points in $K\cdot E\cdot O$, considering all K, must be dense in O.

In order to prove the necessity, let $E\cdot O$ be dense in O and suppose the set $\mathcal{L}$ to be of the second category. On each L there is a vertical interval which is in O and which contains no points of E, that is, it contains only points of CE. By Lemma 2, CE must contain a grating which is in O. But this grating can have within it no point of E since horizontal sections of E are I-sets. This contradicts the hypothesis that E is dense in O, and the theorem is proved.

THEOREM 16. *If vertical sections of R are open, horizontal sections of T are I-sets and $R\cdot T=0$, then $R\cdot T'$ lies on a set $\mathcal{K}$ of vertical sections, which is of the first category.*

Suppose that $\mathcal{K}$ is of the second category. Each $K\epsilon\mathcal{K}$ contains a vertical interval with a point of $R\cdot T'$ as center and containing only points of R. By Lemma 2, there is a grating, composed of points of R, containing a point of $R\cdot T'$ on its interior. This is impossible because horizontal sections of T are I-sets.

COROLLARY 3. *If vertical sections of E are closed and horizontal sections of E are I-sets, then $(CE)\cdot E'$ lies on a set $\mathcal{K}$, of vertical sections, which is of the first category.*

This corollary may be proved by replacing R and T by CE and E in Theorem 16.

If there exists a set $\mathcal{K}$ of vertical sections, and a set $\mathcal{L}$ of horizontal sections, so that every point of E lies either on a member of $\mathcal{K}$ or a member of $\mathcal{L}$, the set E is said to lie on the set $\mathcal{K}$ *plus* the set $\mathcal{L}$. This language is used to distinguish this case from the case in which every point of E lies both on a member of $\mathcal{K}$ and on a member of $\mathcal{L}$; in this latter case E is said to lie on the set $\mathcal{K}$ *and* on the set $\mathcal{L}$.

COROLLARY 4. *If vertical sections of R are open, horizontal sections of T are open and* $R\cdot T=0$, *then* $R'\cdot T+R\cdot T'$ *lies on a set* $\mathcal{K}$ *of vertical sections plus a set* $\mathcal{L}$ *of horizontal sections, both* $\mathcal{K}$ *and* $\mathcal{L}$ *being of the first category.*

This is a slightly stronger conclusion than that of Theorem 13, made possible by the stronger hypothesis given here.

COROLLARY 5. *If vertical and horizontal sections of both R and T are open, and* $R\cdot T=0$, *then* $R'\cdot T+R\cdot T'$ *lies on a set* $\mathcal{K}$ *of vertical sections and a set* $\mathcal{L}$ *of horizontal sections, both* $\mathcal{K}$ *and* $\mathcal{L}$ *being of the first category.*

In Corollary 5, the projection of $R'\cdot T+R\cdot T'$ on any horizontal or vertical section must be of the first category in the section. This is not necessarily true in Corollary 4.

It will be assumed in the following theorem that the space A is dense in itself in order that the set $\overline{E}$ there considered may be perfect. It will also be assumed that A and B have the properties necessary to apply Theorem 10.

A point of closure of a set E is a point in some neighborhood of which E is closed.

THEOREM 17. *If horizontal and vertical sections of E are closed and each point of E lies on a closed horizontal interval containing only points of E, points of closure of E in E are dense on E.*

By Theorem 10, E is an F_σ. It must then be an F_σ in $\overline{E}=E+E'$.† It is necessary to prove that $\overline{E}-E$ is nowhere dense in $\overline{E}$ or, in other words, that limit points of E not in E are not dense on E. It will be shown that $\overline{E}-E$ is of the first category. By Corollary 3, $\overline{E}-E$ (which is the same as $(CE)\cdot E'$) lies on a set $\mathcal{K}$, of vertical sections, of the first category. Let A_0 be the set of points in which the sections of $\mathcal{K}$ cut A. The set $A_0=\sum A_i$ where each A_i is nowhere dense in A. Let R_i be the points of $\overline{E}-E$ lying on those sections, of the set $\mathcal{K}$, which cut A in A_i. If R_i were dense on some portion of $\overline{E}$, it would have to have as a limit point every point of some horizontal interval. This follows from the hypothesis that every point of E lies on a horizontal interval. This is impossible since the projection, A_i, of R_i on A, would then be everywhere dense in some open set in A. Therefore R_i is nowhere dense in $\overline{E}$, and $\overline{E}-E=\sum R_i$ is of the first category in $\overline{E}$. It follows that $\overline{E}-E$ is nowhere dense in $\overline{E}$, for $\overline{E}-E$ is a G_δ and if a G_δ is of the first category, it is nowhere dense.‡

† In this particular case $\overline{E}=E'$.

‡ For example see Blue, Mathematische Annalen, vol. 102, p. 627, in the proof of Theorem 1.

8. Applications

The spaces A and B are restricted here in the same manner as in §7. An interesting application of Corollary 5 is in the proof of the following result of Baire†:

If $f(x, y)$ is a real-valued function defined on the space $A \times B$ and is continuous in each of the variables separately, points of discontinuity of $f(x, y)$ lie on a set of vertical sections *and* a set of horizontal sections, both sets of sections being of the first category.

Let (r_i) be the set of rational numbers. Let R_i be the points of $A \times B$ at which $f(x, y) > r_i$ and let T_j be the set at which $f(x, y) < r_j$. From the properties of continuous functions, R_i and T_j have open horizontal and vertical sections and are disjoined. It follows that $R_i' \cdot T_j + R_i \cdot T_j'$ lies on a set of horizontal *and* a set of vertical sections of the first category, and the same is true of $\sum_{ij}(R_i' \cdot T_j + R_i \cdot T_j')$, the sum being taken only over pairs of i and j for which $r_j < r_i$. The points of this sum are the points of discontinuity of $f(x, y)$.

Applying Corollary 4 in the same way to a function upper semi-continuous in one variable and lower semi-continuous in the other, it may be shown that the set of discontinuities, E, of such a function lies on a set of horizontal sections *plus* a set of vertical sections of the first category. The set E is of the first category, but not all sets of the first category lie on a set of horizontal *plus* a set of vertical sections of the first category. This conclusion therefore contains a result not given by Kempisty.‡

---

† Acta Mathematica, 1899, p. 94.

‡ Fundamenta Mathematicae, vol. 14, p. 237.

University of Iowa,
Iowa City, Iowa